

Geometric numerical integrators for Hunter–Saxton-like equations

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Abstract

We present novel geometric numerical integrators for Hunter–Saxton-like equations by means of new multi-symplectic formulations and known Hamiltonian structures of the problems. We consider the Hunter–Saxton equation, the modified Hunter–Saxton equation, and the two-component Hunter–Saxton equation. Multi-symplectic discretisations based on these new formulations of the problems are exemplified by means of the explicit Euler box scheme, and Hamiltonian-preserving discretisations are exemplified by means of the discrete variational derivative method. We explain and justify the correct treatment of boundary conditions in a unified manner. This is necessary for a proper numerical implementation of these equations and was never explicitly clarified in the literature before, to the best of our knowledge. Finally, numerical experiments demonstrate the favourable behaviour of the proposed numerical integrators.

1 Introduction

Since its introduction in the seminal paper [14], the Hunter–Saxton equation (HS equation below)

$$u_{xxt} + 2u_x u_{xx} + uu_{xxx} = 0, \quad x \in \mathbb{R}, t \geq 0 \quad (1)$$

where $u := u(x, t)$, has been attracting much attention. This is mainly due to its rich mathematical structures: the Hunter–Saxton equation is integrable; it is bihamiltonian; it possesses a Lax pair; it does not have global smooth solutions but enjoy two distinct classes of global weak solutions (conservative and dissipative); it can be seen as the geodesic equation of a right-invariant metric on a certain quotient space; etc. [16, 17, 1, 19] and references therein. Furthermore, the Hunter–Saxton equation arises as a model for the propagation of weakly nonlinear orientation waves in a nematic liquid crystal [14] and it can be seen as the high frequency limit of another well known and well studied equation, namely the Camassa–Holm equation. More about this last equation can be found, for example, in the work [30], the recent review [13], and references therein.

Furthermore, the following extensions of the Hunter–Saxton equation enjoy intense ongoing research activities: the modified Hunter–Saxton equation (mHS equation below), introduced in [20],

$$u_{xxt} + 2u_x u_{xx} + uu_{xxx} - 2\omega u_x = 0, \quad (2)$$

where $\omega > 0$, and the two-component Hunter–Saxton system (2HS below), introduced in [32],

$$\begin{aligned} u_{xxt} + 2u_x u_{xx} + uu_{xxx} - \kappa \rho \rho_x &= 0, \\ \rho_t + (u\rho)_x &= 0, \end{aligned} \quad (3)$$

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where $\kappa \in \{-1, 1\}$ and $\rho := \rho(x, t)$. These two partial differential equations (PDEs) also possess many interesting properties. The modified Hunter–Saxton equation is a model for short capillary waves propagating under the action of gravity [21]. An interesting feature of this modified version of the original problem is that it admits (smooth as well as cusped) travelling waves. This is not the case for the original problem (1). Moreover, this PDE is also bihamiltonian [21]. The two-component generalisation of the Hunter–Saxton equation is a particular case of the Gurevich–Zybin system which describes the dynamics in a model of non-dissipative dark matter, see [28] and also [24]. As the original equation, this system is integrable; has a Lax pair; is bihamiltonian; it is also the high-frequency limit of the two-component Camassa–Holm equation; has peakon solutions; its flow is equivalent to the geodesic flow on a certain sphere; etc. [33, 18, 26, 31, 22] and references therein.

Despite the fact that the above nonlinear PDEs are relatively well understood in a more theoretical way, there are not much results on numerical discretisations of these problems. In fact, although we still continuously find a number of papers on theoretical aspects every year as of writing this paper, we are only aware of the numerical schemes from [12, 34, 35], the latest being proposed in 2010. All the three schemes are only for the original Hunter–Saxton equation. The work [12] proves convergence of some discrete finite difference schemes to dissipative solutions of the Hunter–Saxton equation on the half-line. The references [34, 35] analyse local discontinuous Galerkin methods for the Hunter–Saxton equation and in particular, using results from [12], prove convergence of the discretisation scheme to the dissipative solutions.

The reason for such a lack of numerical studies should be attributed to the following points. First, due to the mixed derivatives present in the above problems, standard spatial discretisations of these nonlinear partial differential equations become, in general, nontrivial. Second, partially in connection with this, the Hunter–Saxton-like equations are essentially underdetermined, and some strong (sometimes exotic) assumptions are necessary to make the solution unique, which causes difficulties in implementing numerical schemes. In fact, in the existing numerical studies described above, some additional boundary conditions are employed to determine the numerical solution without any systematic justification.

In the present publication, we will clarify the above issue and investigate two typical domain settings in details: the half real line and the periodic cases. We provide a systematic view on the necessary and associated additional boundary conditions under which the solution is determined uniquely. This not only gives a clear explanation for the additional conditions employed in the existing schemes, but also provides a basis for the new schemes in the present paper.

Based on this, the main goal of this article is to present novel geometric numerical integrators for the Hunter–Saxton equation and for its two generalisations. As seen above, these nonlinear PDEs have important applications in physical sciences. One class of the proposed numerical schemes is based on new multi-symplectic formulations of the problems and are thus specially designed to preserve the multi-symplectic structures of the original equations. In addition, it was observed in the literature that multi-symplectic integrators have excellent potential for capturing long time dynamics of PDEs. Furthermore, the multi-symplectic schemes for the Hunter–Saxton like equations presented in this article are explicit integrators. The other class of schemes is based on Hamiltonian-preserving discretisations and are thus energy-preserving by construction. Most of these numerical schemes are implicit. A convergence analysis of the proposed numerical schemes will be reported elsewhere.

We also illustrate the validity of the proposed schemes in some numerical examples. More specifically, in this paper we focus on a typical exact solution of the Hunter–Saxton equation that lacks smoothness, and also several travelling waves for the other equations. For both cases geometric numerical integrators are preferable; for the former, geometric numerical integrators produce stable solutions without some stabilization such as upwinding (we will show a comparison in Section 3.1), and for the latter geometric numerical integrators have an advantage in terms of long time behaviour.

The rest of the paper is organised as follows. We discuss the treatment of boundary conditions in the continuous setting with emphasis on how we can determine solutions in the Hunter–Saxton-like equations uniquely in Section 2. Then we present multi-symplectic formulations and numerical schemes for the considered class of PDEs in Section 3. The presentation of Hamiltonian-preserving numerical integrators is done in Section 4.

Finally, we draw some conclusions in Section 5.

We will make use of the following notation. Spatial indices are denoted by n and temporal ones by i . Numerical solutions are denoted by $u^n \simeq u(x_n, \cdot)$, $u^i \simeq u(\cdot, t_i)$ or $u^{n,i} \simeq u(x_n, t_i)$ on a uniform rectangular grid. We set $\Delta x = x_{n+1} - x_n$, $n \in \mathbb{Z}$, and $\Delta t = t_{i+1} - t_i$, $i \geq 0$. When we consider the domain $[-L, L]$, we set $x_0 = -L, x_N = L$. When we consider periodic boundary conditions on the domain $[0, L]$, we set $x_0 = 0, x_N = L$, and assume $u^{n+N} = u^n$ as usual. We often write a solution as a vector $\mathbf{u}^i = (\dots, u^{0,i}, u^{1,i}, u^{2,i}, \dots)$ and use the abbreviation $\mathbf{u}^{i+1/2} = (\mathbf{u}^{i+1} + \mathbf{u}^i)/2$. We define the forward and backward differences in time

$$\delta_t^+ u^{n,i} = \frac{u^{n,i+1} - u^{n,i}}{\Delta t} \quad \text{and} \quad \delta_t^- u^{n,i} = \frac{u^{n,i} - u^{n,i-1}}{\Delta t},$$

and similarly for differences in space. Also, we shall need the centered differences $\delta_t = \frac{1}{2}(\delta_t^+ + \delta_t^-)$ and $\delta_x = \frac{1}{2}(\delta_x^+ + \delta_x^-)$. We also define the operator $\tilde{\delta}_x^2$ by

$$\tilde{\delta}_x^2 u^{n,i} = \frac{u^{n+1,i} - 2u^{n,i} + u^{n-1,i}}{\Delta x^2}.$$

Note that $\delta_x^2 \neq \tilde{\delta}_x^2$. The trapezoidal rule is used as a discretisation of integrals:

$$\sum_{n=0}^N {}'' f^n \Delta x = \left(\frac{1}{2} f^0 + \sum_{n=1}^{N-1} f^n + \frac{1}{2} f^N \right) \Delta x.$$

Finally, we will need the summation-by-parts formula

$$\sum_{n=0}^N {}'' f^n (\delta_x^+ g^n) \Delta x + \sum_{n=0}^N {}'' (\delta_x^- f^n) g^n \Delta x = \left[\frac{f^n g^{n+1} + f^{n-1} g^n}{2} \right]_0^N$$

which is frequently used for the analyses of Hamiltonian-preserving schemes.

2 On difficulties in solving numerically Hunter–Saxton-like equations: the treatment of boundary conditions

As mentioned in Section 1, many difficulties arise in solving Hunter–Saxton-like (HS-like) equations numerically. Briefly speaking, solutions of HS-like equations are not unique due to the operator ∂_x^2 in front of u_t , and thus even in the continuous setting it is a hard task to determine how we should “choose” the solution. Furthermore, even if it is done in the continuous case, different difficulties arise in the discrete setting where we are forced to impose additional discrete boundary conditions (this will be illustrated and discussed in detail in the subsequent sections.)

In this section, we consider two typical domain settings, the half real line case and the periodic case, and describe the situation in detail in the continuous case.

In preparation for the following discussion, let us first precisely consider the meaning of “underdetermined.” Integrating both sides of the HS equation (1) twice on some spatial domain, say $[-L, L]$ in view of numerical computation, we obtain

$$u_{xtt} + 2u_x u_{xx} + uu_{xxx} = 0, \tag{4}$$

$$(u_t + uu_x)_x - \frac{1}{2} u_x^2 = a(t), \tag{5}$$

$$u_t + uu_x - \partial_x^{-1} \left(\frac{1}{2} u_x^2 + a(t) \right) = h(t), \tag{6}$$

where $\partial_x^{-1}(\cdot) := \int_{-L}^x (\cdot) dx$, and $a(t)$ and $h(t)$ are the integral constants (independent of x). We refer to (4), (5) and (6) as “rank -2 ”, “rank -1 ” and “rank 0 ” equations, respectively (“rank” denotes -1 times the order of spatial differentiation of u_t).

Consider the rank 0 equation (6). It is reasonable to assume that the solution is unique for given $a(t)$ and $h(t)$ under appropriate boundary conditions. If the constants and boundary conditions are given in advance, one can discretise the rank 0 equation. However, if one considers the discretisation based on the lower rank equations a difficulty arises: solutions to the lower rank equations under the same boundary conditions as for the rank 0 equation cannot be determined uniquely because of the lack of information on either $a(t)$ or $h(t)$. Hence, we must introduce some additional assumptions to recover the lost information. In what follows, we show how this can be done.

2.1 The half line case

The Hunter–Saxton equation (1) was originally proposed as the rank -1 equation with $a(t) = 0$ on the whole real line [14]. There, with the method of characteristics, it was shown that we need one boundary condition “ $u(x, t) \rightarrow 0$ as $x \rightarrow \infty$ ” to determine a solution. Similarly, a solution can be determined under the boundary condition $u(x, t) \rightarrow 0$ as $x \rightarrow -\infty$. An exact (weak) solution can be constructed in closed form:

$$u(x, t) = \begin{cases} 0 & \text{if } x \leq 0 \\ x/(0.5t + 1) & \text{if } 0 < x < (0.5t + 1)^2 \\ 0.5t + 1 & \text{if } x \geq (0.5t + 1)^2. \end{cases} \quad (7)$$

See Figure 1 for a snapshot. In view of this solution, one typical setting of the Hunter–Saxton equation is that we consider the equation on the half real line $[0, \infty)$, and we impose $u(0, t) = 0$. In this case, the unknown constant $h(t)$ can be easily found as $h(t) = 0$ by considering the rank 0 equation at $x = 0$, which is consistent with the claim that the solution can be uniquely determined. Many studies that follows [14], including the numerical studies [12, 34, 35], inherit this setting.

In the numerical computations, however, we have to “cut” the half line to a bounded domain, which we assume $[-L, L]$ without loss of generality. Then it is natural to impose $u(-L, t) = 0$ ($t > 0$), which corresponds to $u(x, t) \rightarrow 0$ as $x \rightarrow -\infty$.

Furthermore, when we consider finite difference discretisations, we discretise the equations uniformly in the spatial domain, and thus the schemes use grid points outside the spatial domain. Discrete boundary conditions for solving PDEs numerically are usually set such that they are consistent with the provided continuous boundary conditions. This task is not straightforward for the HS equation (1), since we generally need boundary conditions at both the left and right boundaries, while in the original half line setting there is only one boundary condition $u(-L, t) = 0$ (see also Remark 1). Below we explain and justify the treatment in the existing numerical studies, and how we can justify it.

First, in view of the exact solution above, it is customary to impose $u_x(L, t) = 0$, which is safe for times t in the time interval $[0, 2(\sqrt{L} - 1))$; i.e., we understand that we take L large enough so that the additional condition is safe for times we are interested in.

Depending on the finite difference scheme we employ, we often have to seek further additional conditions. Following the existing studies and also motivated by the discussion on the exact solution above, let us focus on the solutions under the boundary conditions $u(-L, t) = u_x(L, t) = 0$ and with the constants $a(t) = h(t) = 0$. Note that $u_x(L, t) = 0$ corresponds to $u_x(x, t) \rightarrow 0$ as $x \rightarrow \infty$ (for the theoretical studies under the condition $u_x(x, t) \rightarrow 0$ ($x \rightarrow \infty$), see, e.g. Zhang–Zheng [37]). We now assume that the solution of the rank 0 equation (6) in this problem setting is unique for each initial condition. Below we show that this unique solution automatically satisfies $u_x(-L, t) = u_{xx}(L, t) = 0$:

- $u_x(-L, t) = 0$:

Let $v(\cdot) := u_x(-L, \cdot)$. By evaluating (5) at $x = -L$, we obtain $\dot{v} + \frac{1}{2}v^2 = 0$. In general, a solution of

this ODE is of the form $v(t) = 2/(t + c_1)$ with a constant c_1 , but as long as we consider an initial value satisfying $v(0) = 0$ (in Section 3, we considered such a case), we have $v(t) = 0$ along the solution.

- $u_{xx}(L, t) = 0$:

Let $w(\cdot) := u_{xx}(L, \cdot)$. By evaluating (5) at $x = L$, we obtain $uu_{xx}|_{x=L} = 0$. Here we exclude the case $u(L, t) = 0$, because this will give the solution $u(x, t) = 0$. This can be understood by (6), noting $\int_{-L}^L u_x^2 dx = 0$. Therefore we immediately obtain $u_{xx}(L, t) = 0$.

These additional conditions were employed in the previous numerical studies [12, 34, 35], and are also used for geometric numerical integrators proposed in this paper. The number of additional conditions depends on each scheme.

Remark 1. If we carefully discretise the rank -1 equation, it is possible to construct a scheme that happily works with only one boundary condition $u(-L, t) = 0$, which is consistent with the continuous case. As far as the present authors understand, it has never been pointed out explicitly in the literature. Such a scheme will be reported elsewhere since its exposition is outside the scope of this paper.

2.2 The periodic case

First of all, we note an important fact that in the periodic case the Hunter–Saxton-like equations are essentially underdetermined in rank -1 (and accordingly rank -2), since there is no way to “add” additional boundary conditions. The only way to determine a solution is to provide the unknown constants $a(t)$ and $h(t)$; see, for example, Yin [36], where the author considered the rank -1 HS with $a(t) = -(1/L) \int_0^L u_x^2/2 dx$ (which is necessary such that $u(x, t)$ is actually periodic) and showed an unique existence theorem under the assumption that $h(t)$ is explicitly given. Thus one way to consider numerical computation is that we seek a scheme that somehow incorporates the given information $h(t)$.

In the present paper, however, let us consider a different situation. The mHS and 2HS equations have smooth travelling wave solutions, which is in sharp contrast to the HS where any strong solutions blow up in finite time [36]. Thus it makes sense to focus on such waves in the mHS and 2HS.

For the mHS equation, we focus on the smooth travelling waves of speed c , $u(x, t) = \varphi(x - ct)$, which are solutions to the differential equation [21]

$$(\varphi')^2 = \frac{2\omega(M - \varphi)(\varphi - m)}{c - \varphi},$$

where M , resp. m , is the maximum, resp. minimum, of the wave whenever $m < M < c$. The period of the wave is denoted by L , and the spatial domain is set to $[0, L]$.

Now we illustrate how we can “recover” the unknown constant $h(t)$ for such a travelling wave. To see this, we first need to understand again why solutions of the mHS equation (2), under periodic boundary conditions, are not unique. We proceed as for the HS equation. Integrating both sides of the rank -2 equation (2), we obtain the rank -1 equation

$$(u_t + uu_x)_x - \frac{1}{2}u_x^2 - 2\omega u - a(t) = 0 \quad (8)$$

with a constant $a(t)$. As long as we consider periodic boundary conditions, $a(t)$ has a unique expression of the form

$$a(t) = -\frac{1}{L} \int_0^L \left(\frac{1}{2}u_x^2 + 2\omega u \right) dx,$$

because

$$0 = \int_0^L \left((u_t + uu_x)_x - \frac{1}{2}u_x^2 - 2\omega u - a(t) \right) dx = - \int_0^L \left(\frac{1}{2}u_x^2 + 2\omega u \right) dx - La(t).$$

Furthermore, integrating (8) once again, we obtain the rank 0 equation

$$u_t + uu_x - \partial_x^{-1} \left(\frac{1}{2} u_x^2 + 2\omega u + a(t) \right) = (u_t + uu_x)|_{x=0}$$

where $\partial_x^{-1}(\cdot) := \int_0^x (\cdot) dx$. By introducing $h(t) := (u_t + uu_x)|_{x=0}$, the above equation can be rewritten as

$$u_t + uu_x - \partial_x^{-1} \left(\frac{1}{2} u_x^2 + 2\omega u + a(t) \right) = h(t). \quad (9)$$

Below we show an interesting fact that, as far as we are concerned with travelling waves under the periodic boundary condition, the constant $h(t)$, and accordingly the uniqueness of the solution can be easily and automatically recovered by the concept of the pseudo-inverse of the differential (or difference) operator. Although pseudo-inverses are quite common and in fact often used in numerical analysis, it seems it has never been pointed out in the literature that pseudo-inverses can be effectively utilised in such a way.

Since the concrete form of the mHS equation (9) is not essential for this discussion, we consider partial differential equations of the general form

$$u_t + f(u, u_x, u_{xx}, \dots) = h(t), \quad (10)$$

as an equation of rank 0 (below we often use the abbreviation $f(u) = f(u, u_x, u_{xx}, \dots)$). Assume that (10) has a periodic travelling wave solution $u(x, t) = \varphi(x - ct)$ with $\varphi(x) = \varphi(x + L)$, where L denotes the length of the period. Substituting φ into (10), we obtain

$$-c\varphi_x + f(\varphi, \varphi_x, \dots) = h(t).$$

For any g satisfying $[g(\varphi)]_0^L = 0$, one can re-express $h(t)$ as follows:

$$\begin{aligned} h(t) &= \frac{\int_0^L g'(\varphi) dx}{\int_0^L g'(\varphi) dx} h(t) + \frac{\int_0^L g'(\varphi) c\varphi_x dx}{\int_0^L g'(\varphi) dx} \\ &= \frac{\int_0^L g'(\varphi) (h(t) + c\varphi_x) dx}{\int_0^L g'(\varphi) dx} = \frac{\int_0^L g'(\varphi) f(\varphi) dx}{\int_0^L g'(\varphi) dx}. \end{aligned}$$

As a special case if we select $g'(\varphi) = 1$ (i.e., $g(\varphi) = \varphi$), we have

$$h(t) = \frac{\int_0^L f(\varphi) dx}{L}. \quad (11)$$

In short, if the equation (10) has a periodic travelling wave solution, $h(t)$ should be expressed as (11).

Let us now consider the situation where one does not know $h(t)$ explicitly and thus one is forced to work on equations of lower ranks, for example,

$$u_{tx} + f_x(u, u_x, u_{xx}, \dots) = 0.$$

Let us introduce a pseudo-inverse operator of ∂_x , denoted by ∂_x^\dagger (one can define a pseudo-inverse operator for a closed linear operator between two Hilbert spaces, see [10, 11] for example), and consider the following PDE

$$u_t + \partial_x^\dagger f_x(u, u_x, u_{xx}, \dots) = 0. \quad (12)$$

Proposition 1. *The partial differential equation (12) is equivalent to*

$$u_t + f(u, u_x, u_{xx}, \dots) = \frac{\int_0^L f(u) dx}{L}. \quad (13)$$

This proposition indicates that the problem (12) automatically catches travelling waves whenever they exist. Thus, for a proper numerical discretisation of travelling waves of the mHS equation, one should use the formulation (12).

Proof. Since $\text{Ker}(\partial_x) = \{v \mid v = \text{const.}\}$, equation (12) can be rewritten as

$$u_t + f(u, u_x, u_{xx}, \dots) = k(t),$$

where $k(t)$ is a constant independent of x and minimises $\|u_t\|_{L^2}$. Since

$$\begin{aligned} \int_0^L u_t^2 dx &= \int_0^L (k(t) - f(u))^2 dx = Lk(t)^2 - 2 \left(\int_0^L f(u) dx \right) k(t) + \int_0^L f(u)^2 dx \\ &= L \left(k(t) - \frac{\int_0^L f(u) dx}{L} \right)^2 - \frac{\left(\int_0^L f(u) dx \right)^2}{L} + \int_0^L f(u)^2 dx, \end{aligned}$$

$k(t)$ should be of the form

$$k(t) = \frac{\int_0^L f(u) dx}{L}.$$

This concludes the proof of the proposition. □

A similar discussion for equations of rank -2 shows that the problem

$$u_t + (\partial_x^2)^\dagger f_{xx}(u, u_x, u_{xx}, \dots) = 0$$

is equivalent to (13).

In this paper, we also consider the periodic 2HS equation (3) and its travelling waves. When $\kappa = 1$, the smooth periodic travelling waves of speed c , $u(x, t) = \varphi(x - ct)$, are solutions to the differential equation [23]

$$(\varphi')^2 = \frac{b(Z - \varphi)(\varphi - z)^2}{c - \varphi}, \quad (14)$$

where $Z := \max_{y \in \mathbb{R}} \varphi(y)$, resp. $z := \min_{y \in \mathbb{R}} \varphi(y)$, is the maximum, resp. minimum, of the wave. Here, b is an additional positive parameter. Furthermore,

$$\rho(x, t) = \psi(x - ct) = \frac{a}{c - \varphi}, \quad (15)$$

where the parameter a is determined by b, z, Z and c , i.e. $a = \sqrt{b(c - z)(c - Z)}$. Similar to the discussion about the periodic mHS equation seen above, one can show that a correct simulation of the travelling waves for the periodic 2HS equation should be based on the following differential equation with the pseudo-inverse operator $(\partial_x^2)^\dagger$:

$$\begin{aligned} u_t + (\partial_x^2)^\dagger (2u_x u_{xx} + uu_{xxx} - \kappa \rho \rho_x) &= 0, \\ \rho_t + (u\rho)_x &= 0. \end{aligned}$$

3 Multi-symplectic integrations of Hunter–Saxton-like equations

We shall first present two new multi-symplectic formulations of the Hunter–Saxton equation, the corresponding explicit multi-symplectic schemes, as well as numerical experiments supporting our theoretical findings in Subsection 3.1. Subsection 3.2 gives a similar program for the multi-symplectic discretisation of the modified Hunter–Saxton equation. Using ideas from the above mentioned subsections, we shall present multi-symplectic schemes for the two-component Hunter–Saxton equation in Subsection 3.3.

The proposed multi-symplectic formulations of the HS-like equations follow the one presented in [6] for the Camassa–Holm equation and in [5] for the two-component Camassa–Holm equation.

For a detailed exposition on the concept of multi-symplectic partial differential equations, we refer the reader to, for example, the early references [25, 2, 3].

3.1 Multi-symplectic discretisations of the Hunter–Saxton equation (half line case)

We consider the numerical discretisation of the Hunter–Saxton (HS) equation (1) in a computational domain $[-L, L]$ for a positive real number L (correspondingly, we set $x_0 = -L$ and $x_N = L$ with a natural number N). In this subsection, we consider the boundary conditions given by $u(-L, t) = u_x(-L, t) = u_x(L, t) = 0$ (see the previous section for the consistency of these conditions).

3.1.1 First multi-symplectic formulation and integrator for HS

The multi-symplectic formulation

$$M z_t + K z_x = \nabla_z S(z) \quad (17)$$

of (1) is obtained with $z = [u, \phi, w, v, \eta]^T$, the gradient of the scalar function $S(z) = -wu - u\eta^2/2 + \eta v$ and the two skew-symmetric matrices

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For convenience, we also write this system componentwise

$$\begin{aligned} -\frac{1}{2}\eta_t - v_x &= -w - \frac{1}{2}\eta^2, \\ w_x &= 0, \\ -\phi_x &= -u, \\ u_x &= \eta, \\ \frac{1}{2}u_t &= -u\eta + v. \end{aligned}$$

A key observation, [3], for the above multi-symplectic formulation of our problem is that the two skew-symmetric matrices M and K define symplectic structures on subspaces of \mathbb{R}^5

$$\omega = dz \wedge M dz, \quad \zeta = dz \wedge K dz,$$

thus resulting in the following multi-symplectic conservation law

$$\partial_t \omega + \partial_x \zeta = 0. \quad (18)$$

This is a local property of our problem and we thus hope that multi-symplectic numerical schemes, as derived soon, will render well local properties of the HS equation (1). More explicitly, we have for any solutions of (17), the local conservation laws

$$\partial_t E(z) + \partial_x F(z) = 0 \quad \text{and} \quad \partial_t I(z) + \partial_x G(z) = 0,$$

with the density functions

$$\begin{aligned} E(z) &= S(z) - \frac{1}{2} z_x^T K^T z, & F(z) &= \frac{1}{2} z_t^T K^T z, \\ G(z) &= S(z) - \frac{1}{2} z_t^T M^T z, & I(z) &= \frac{1}{2} z_x^T M^T z. \end{aligned}$$

Under the usual assumption on vanishing boundary terms for the functions $F(z)$ and $G(z)$ one obtains the following global conserved quantities

$$\mathcal{E}(z) = \int_{-L}^L E(z) dx \quad \text{and} \quad \mathcal{I}(z) = \int_{-L}^L I(z) dx. \quad (19)$$

For our choice of the skew-symmetric matrices M and K , one thus obtains the density functions

$$\begin{aligned} E(z) &= S(z) + \frac{1}{2} z_x^T K z = \frac{1}{2} (-wu - u(u_x)^2 + (uv)_x), \\ F(z) &= -\frac{1}{2} z_t^T K z = \frac{1}{2} u_t v - \frac{1}{2} \phi_t w + \frac{1}{2} w_t \phi - \frac{1}{2} v_t u, \\ G(z) &= S(z) + \frac{1}{2} z_t^T M z = -wu - \frac{1}{2} \eta^2 u + v\eta - \frac{1}{4} \eta u_t + \frac{1}{4} \eta_t u, \\ I(z) &= -\frac{1}{2} z_x^T M z = \frac{1}{4} (u_x \eta - \eta_x u). \end{aligned}$$

This will help us to derive the corresponding global invariants (19).

We first integrate the local conservation law $\partial_t I(z) + \partial_x G(z) = 0$ over the spatial domain and obtain the Hamiltonian

$$\mathcal{H}_1(u, u_x) = \frac{1}{2} \int_{-L}^L u_x^2 dx. \quad (20)$$

Indeed for the above local conservation law, one has

$$0 = \frac{1}{4} \frac{d}{dt} \int_{-L}^L (u_x^2 - uu_{xx}) dx + \left[-\frac{3}{4} u_{xt} u + \frac{1}{4} u_t u_x - u^2 u_{xx} \right]_{-L}^L.$$

Using one integration by parts and the vanishing boundary conditions for u one thus gets

$$\frac{1}{2} \frac{d}{dt} \int_{-L}^L u_x^2 dx = 0$$

which after integration gives the Hamiltonian (20).

Similarly, the second conservation law $\partial_t E(z) + \partial_x F(z) = 0$ is linked to the Hamiltonian

$$\mathcal{H}_2(u, u_x) = \frac{1}{2} \int_{-L}^L uu_x^2 dx$$

but, for our boundary conditions, this expression is not constant along solutions to our problem

$$\frac{d}{dt} \mathcal{H}_2(u, u_x) = \frac{1}{2} u_t^2 \Big|_{x=L}.$$

Indeed, noting that $w \equiv 0$ so that the multi-symplectic formulation (17) is equivalent to the rank -1 equation with $a(t) = 0$, one can simplify the density functions as follows

$$\begin{aligned} E(z) &= -\frac{uu_x^2}{2} + \frac{1}{2} (uv)_x, \\ F(z) &= \frac{1}{2} u_t v - \frac{1}{2} v_t u. \end{aligned}$$

Integrating now the second conservation law, one gets

$$\begin{aligned} 0 &= -\frac{d}{dt} \mathcal{H}_2(u, u_x) + \frac{d}{dt} \left[\frac{uv}{2} \right]_{-L}^L + \left[\frac{u_t v - v_t u}{2} \right]_{-L}^L \\ &= -\frac{d}{dt} \mathcal{H}_2(u, u_x) + \frac{u_t v + uv_t}{2} \Big|_{x=L} + \frac{u_t v - v_t u}{2} \Big|_{x=L} \end{aligned}$$

which reduces to

$$\frac{d}{dt}\mathcal{H}_2(u, u_x) = \frac{1}{2}u_t^2 \Big|_{x=L}.$$

We now derive a numerical scheme based on the above multi-symplectic formulation of the Hunter–Saxton equation.

Following [27], one may obtain an integrator satisfying a discrete version of the multi-symplectic conservation law (18) by introducing a splitting of the two matrices M and K in (17), setting $M = M_+ + M_-$, $K = K_+ + K_-$ where $M_+^T = -M_-$ and $K_+^T = -K_-$. We will only consider the following matrices $M_+ = \frac{1}{2}M$ and $K_+ = \frac{1}{2}K$ for the splitting, keeping in mind that the above splitting of the matrices is not unique. The corresponding Euler box scheme reads

$$M_+ \delta_t^+ z^{n,i} + M_- \delta_t^- z^{n,i} + K_+ \delta_x^+ z^{n,i} + K_- \delta_x^- z^{n,i} = \nabla_z S(z^{n,i}), \quad (21)$$

where $z^{n,i} \approx z(x_n, t_i)$.

The multi-symplecticity of the Euler box scheme is interpreted in the sense that, recall (18),

$$\delta_t^+ \omega^{n,i} + \delta_x^+ \zeta^{n,i} = 0,$$

where $\omega^{n,i} = dz^{n,i-1} \wedge M_+ dz^{n,i}$ and $\zeta^{n,i} = dz^{n-1,i} \wedge K_+ dz^{n,i}$.

With our choices for the matrices M_+ and K_+ and our multi-symplectic formulation (17) of the HS equation, the centered version of the Euler box scheme (21) reads (expressing the scheme only in the variable $u^{n,i}$)

$$-\delta_x^2 \delta_t u^{n,i} + \frac{1}{2} \delta_x ((\delta_x u^{n,i})^2) - \delta_x^2 (u^{n,i} \delta_x u^{n,i}) = 0.$$

Though the operator δ_x is not invertible, we will work on the following reformulation

$$-\delta_x \delta_t u^{n,i} + \frac{1}{2} (\delta_x u^{n,i})^2 - \delta_x (u^{n,i} \delta_x u^{n,i}) = 0,$$

since it still retains the multi-symplecticity and is consistent with the rank -1 formulation. Taking boundary conditions into account, we finally formulate the Euler box scheme as follows:

$$\begin{aligned} v^{n,i} &:= \delta_x u^{n,i}, \\ \delta_t v^{n,i} &= \frac{1}{2} (v^{n,i})^2 - \delta_x (u^{n,i} v^{n,i}) \end{aligned} \quad (22)$$

for $n = 1, \dots, N-1$, under the boundary conditions $u^{0,i} = 0$, $u^{1,i} = u^{0,i}$, $v^{0,i} = 0$ and $v^{N,i} = 0$. They correspond to $u(-L, t) = 0$, $u_x(-L, t) = 0$, $u_x(-L, t) = 0$ and $u_x(L, t) = 0$, respectively. Let us point out that the numerical scheme (22) is explicit, and since it preserves some geometry of the original PDE, it will perform well in terms of the evolutions of the Hamiltonians as demonstrated in Subsection 3.1.3.

3.1.2 Second multi-symplectic formulation and integrator for HS

As described in [1], see also [6] in the context of the Camassa–Holm equation, in addition to (1) we can also consider the evolution equation satisfied by the energy density $\alpha := u_x^2$. This permits to distinguish between two solutions, see [1] for details. We thus obtain the following system of partial differential equations equivalent to the HS equation

$$\begin{aligned} u_t + uu_x + P_x &= 0, \\ -P_{xx} &= \frac{1}{2}\alpha, \\ \alpha_t + (u\alpha)_x &= 0. \end{aligned}$$

The multi-symplectic formulation of the above system of partial differential equations is obtained setting $z = [u, \beta, w, \alpha, \phi, \gamma, P, r]$,

$$M = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

and considering the scalar function

$$S(z) = -\gamma u + \frac{u^2 \alpha}{2} - \alpha w + r^2.$$

This is equivalent to the following system

$$\begin{aligned} -\frac{1}{2}\beta_t &= -\gamma + u\alpha, & \frac{1}{2}u_t + w_x + P_x &= 0, \\ -\beta_x &= -\alpha, & -\frac{1}{2}\phi_t &= -w + \frac{u^2}{2}, \\ \frac{1}{2}\alpha_t + \gamma_x &= 0, & -\phi_x &= -u, \\ -\beta_x - 2r_x &= 0, & 2P_x &= 2r. \end{aligned}$$

Using the above multi-symplectic formulation of our problem and similarly as before, the centered version of the Euler box scheme (21) for the second multi-symplectic formulation of the HS equation based on the choice of splitting matrices $M_+ = \frac{1}{2}M$ and $K_+ = \frac{1}{2}K$ reads

$$\begin{aligned} \delta_t \alpha^{n,i} + \delta_x (u^{n,i} \alpha^{n,i}) &= 0, \\ -\delta_x^2 P^{n,i} &= \frac{1}{2} \alpha^{n,i}, \\ \delta_t u^{n,i} + \frac{1}{2} \delta_x ((u^{n,i})^2) + \delta_x P^{n,i} &= 0. \end{aligned} \tag{24}$$

We note that the associated global quantities (19) of the second multi-symplectic formulation are \mathcal{H}_2 and \mathcal{H}_3 , see [15] for the definition of \mathcal{H}_3 . But as will be explained soon, this multi-symplectic scheme (24) also offers a good behaviour for the evolution of \mathcal{H}_1 .

3.1.3 Numerical simulations: Multi-symplectic schemes for HS

We now test our multi-symplectic integrators using the exact (nonsmooth) solution (7) to the HS equation on the computational domain $[-6, 6]$ (i.e., $L = 6$) for times $0 \leq t \leq T_{\text{end}} = 0.5$. Analytical values of the Hamiltonians are $\mathcal{H}_1 = 0.5$ and $\mathcal{H}_2 = t/8 + 1/4$.

Figure 1 shows the numerical results obtained by the first multi-symplectic scheme (22) with step sizes $\Delta x = 12/201$ (i.e., $N = 201$) and $\Delta t = 0.01$. The second multi-symplectic scheme (24) offers similar behaviours for both components of the solution and for both Hamiltonians as the first scheme (22). Results for this second numerical scheme are thus not displayed. As also observed for the numerical methods proposed in [34], oscillations are present in $u_x(x, t)$, but the numerical approximation for $u(x, t)$, on compact intervals, is still correct. Moreover, one can observe the excellent evolutions of the Hamiltonians along the numerical solutions offered by the multi-symplectic scheme (22).

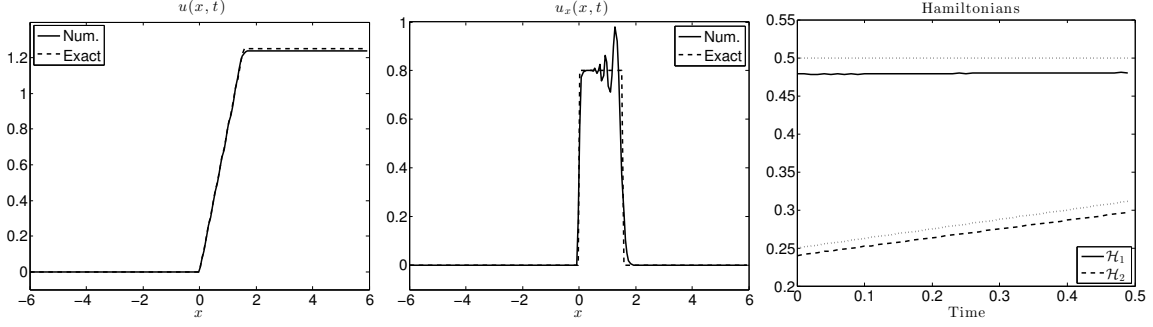


Figure 1: Multi-symplectic scheme (22): exact and numerical profiles of $u(x, t)$ and $u_x(x, t)$ at time $T_{\text{end}} = 0.5$ and computed Hamiltonians ($\Delta x = 12/201$ and $\Delta t = 0.01$).

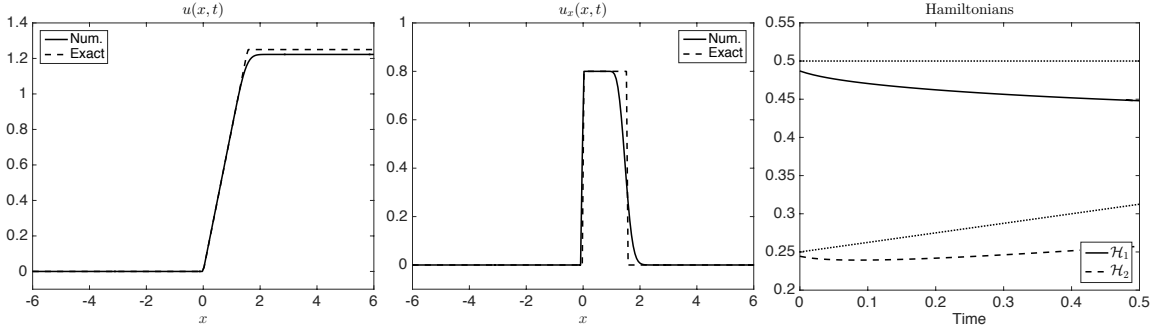


Figure 2: Mid-point discretisation of the semi-discrete scheme by Holden et al. ((3.3) in [12]): exact and numerical profiles of $u(x, t)$ and $u_x(x, t)$ at time $T_{\text{end}} = 0.5$ and computed Hamiltonians ($\Delta x = 12/201$ and $\Delta t = 0.01$).

Figure 2 shows the numerical results obtained by the mid-point discretisation of the semi-discrete scheme proposed by Holden et al. ((3.3) in [12]). We compare our multi-symplectic scheme with this scheme. Though no oscillation is observed for $u_x(x, t)$ in Figure 2, the multi-symplectic scheme (22) seems preferable in terms of approximations of $u(x, t)$ and the Hamiltonians.

3.2 Multi-symplectic integrators for the modified Hunter–Saxton equation (periodic case)

In this section, we consider numerical integrators for the modified Hunter–Saxton (mHS) equation (2). Note that for $\omega = 0$, one obtains the Hunter–Saxton equation (1). An interesting feature of the mHS equation is that it admits (smooth as well as cusped) travelling waves when $\omega > 0$. This is not the case for the original problem (1). The aim of this subsection is thus to derive a multi-symplectic integrator for these travelling waves. Unfortunately, as seen in Section 2, (2) is an “underdetermined” PDE (in the sense that the problem has multiple solutions depending on the constant $h(t)$), which makes the construction of numerical schemes challenging. The results of Section 2 permit to overcome this difficulty and thus to derive a multi-symplectic scheme for the mHS equation. Finally, we present numerical experiments for the travelling waves of the mHS equation.

3.2.1 Multi-symplectic formulation and integrator for mHS

The following multi-symplectic formulation for the mHS equation follows from the one obtained for the HS equation in Subsection 3.1. Indeed, the multi-symplectic formulation (17) of the mHS equation is obtained with $z = [u, \phi, w, v, \eta]^T$, the gradient of the scalar function $S(z) = -wu - u\eta^2/2 + \eta v - \omega u^2$ and the two

skew-symmetric matrices

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For convenience, we also write this system componentwise

$$\begin{aligned} -\frac{1}{2}\eta_t - v_x &= -w - \frac{1}{2}\eta^2 - 2\omega u, \\ w_x &= 0, \\ -\phi_x &= -u, \\ u_x &= \eta, \\ \frac{1}{2}u_t &= -u\eta + v. \end{aligned}$$

As this was done in the previous section, we can integrate the first local conservation law $\partial_t I(z) + \partial_x G(z) = 0$ over the spatial domain and obtain the Hamiltonian

$$\mathcal{H}_1(u, u_x) = \frac{1}{2} \int_0^L u_x^2 dx \quad (25)$$

which is a conserved quantity for the mHS equation. The density functions for the second conservation law $\partial_t E(z) + \partial_x F(z) = 0$ read

$$\begin{aligned} E(z) &= -\frac{wu}{2} + \frac{(uv)_x}{2} - \omega u^2 - \frac{uu_x^2}{2}, \\ F(z) &= \frac{1}{2}(u_t v - \phi_t w + w_t \phi - v_t u). \end{aligned}$$

Integrating this last conservation law gives

$$\begin{aligned} 0 &= -\frac{d}{dt} \int_0^L \left(\frac{uu_x^2}{2} + \omega u^2 \right) dx - \frac{1}{2} \int_0^L (wu)_t dx + \frac{1}{2} \frac{d}{dt} \int_0^L (uv)_x dx \\ &\quad + \frac{1}{2} \int_0^L (u_t v - \phi_t w + w_t \phi - v_t u)_x dx. \end{aligned}$$

Observing that w is constant in x and η, v are periodic, the above equation gives us

$$\frac{d}{dt} \mathcal{H}_2 := \frac{d}{dt} \int_0^L \left(\frac{uu_x^2}{2} + \omega u^2 \right) dx = -w \int_0^L u_t dx$$

for the Hamiltonian

$$\mathcal{H}_2(u, u_x) = \frac{1}{2} \int_0^L (uu_x^2 + 2\omega u^2) dx.$$

Note that the Hamiltonian \mathcal{H}_2 is a conserved quantity for travelling wave solutions but not for general solutions of the mHS equation.

The explicit Euler box scheme (21) for the above multi-symplectic formulation reads

$$-\delta_x^2 \delta_t u^{n,i} + \frac{1}{2} \delta_x ((\delta_x u^{n,i})^2) - \delta_x^2 (u^{n,i} \delta_x u^{n,i}) + 2\omega \delta_x u^{n,i} = 0.$$

As seen in Section 2, in order to select the right travelling waves, we have to consider the pseudo-inverse operator of δ_x^2 , denoted by $(\delta_x^2)^\dagger$. We then obtain the following numerical scheme

$$-\delta_t u^{n,i} + (\delta_x^2)^\dagger \left(\frac{1}{2} \delta_x ((\delta_x u^{n,i})^2) - \delta_x^2 (u^{n,i} \delta_x u^{n,i}) + 2\omega \delta_x u^{n,i} \right) = 0. \quad (26)$$

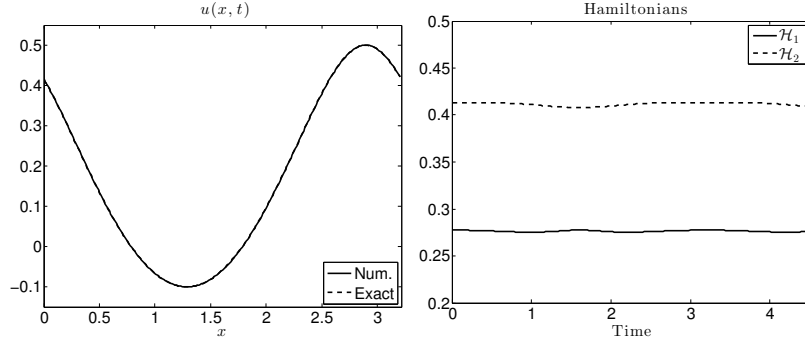


Figure 3: Exact and numerical profiles of $u(x, t)$ at time $T_{\text{end}} = 3.5$ and computed Hamiltonians by the multi-symplectic scheme (26).

3.2.2 Multi-symplectic simulations of travelling waves for mHS

Figure 3 displays the exact and numerical profiles of $u(x, t)$ at time $T_{\text{end}} = 3.5$ and also the computed values of the Hamiltonians

$$\mathcal{H}_1(u, u_x) = \frac{1}{2} \int_0^L u_x^2 dx, \quad \mathcal{H}_2(u, u_x) = \frac{1}{2} \int_0^L (uu_x^2 + 2\omega u^2) dx$$

using the Euler box scheme (26) with (relative large) step sizes $\Delta t = 0.02$ and $\Delta x = L_{\text{per}}/256$. We checked numerically that the solution of the above differential equation for φ is periodic with period $L_{\text{per}} = 3.2151\dots$. The parameters for this simulation on the periodic computational domain $[0, L_{\text{per}}]$ are as follows: $\omega = 1.5, m = -0.1, M = 0.5, c = 1$. One may notice that the numerical solution agrees very well with the exact one. Furthermore, good conservation properties of the numerical scheme are observed, even for such large step sizes, in the present figure.

3.3 Multi-symplectic discretisations of the two component Hunter–Saxton system (periodic case)

In this section we consider the numerical discretisation of the two-component Hunter–Saxton (2HS) system (3) introduced in [32]. This system of partial differential equations also admits travelling wave solutions [23] so that one can use the pseudo-inverse as this was done in the previous subsection. We refer the reader to [23] for an exposition of the Hamiltonian structures of the 2HS system.

3.3.1 Multi-symplectic formulation and integrator for 2HS

We can use the results from the preceding subsections to derive a multi-symplectic formulation for this system of equations too. Indeed, setting $z = [u, \phi, w, v, \eta, \rho, \gamma, \beta]$, and using the following skew-symmetric matrices

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\kappa}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\kappa}{2} & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa \\ 0 & 0 & 0 & 0 & 0 & 0 & -\kappa & 0 \end{bmatrix} \quad (27)$$

and the gradient of the scalar function

$$S(z) = -wu - u\eta^2/2 + \eta v - \kappa u \rho^2/2 + \kappa \gamma \rho$$

one obtains a multi-symplectic formulation (17) for the 2HS system (3). The above formulation reads componentwise:

$$\begin{aligned} -\frac{1}{2}\eta_t - v_x &= -w - \frac{1}{2}\eta^2 - \frac{\kappa}{2}\rho^2, \\ w_x &= 0, \\ -\phi_x &= -u, \\ u_x &= \eta, \\ \frac{1}{2}u_t &= -u\eta + v, \\ \frac{\kappa}{2}\beta_t &= \kappa\gamma - \kappa u \rho, \\ \kappa\beta_x &= \kappa\rho, \\ -\frac{\kappa}{2}\rho_t - \kappa\gamma_x &= 0. \end{aligned}$$

The density functions used to compute the local conservation laws for 2HS are given by

$$\begin{aligned} E(z) &= -\frac{1}{2}uw - \frac{uu_x^2}{2} + \frac{1}{2}u_x v - \frac{\kappa}{2}u\rho^2 + \frac{\kappa}{2}\gamma\rho - \frac{1}{2}w_x\phi + \frac{1}{2}v_x u + \frac{\kappa}{2}\gamma_x\beta, \\ F(z) &= \frac{1}{2}(u_t v - \phi_t w + w_t \phi - uv_t - \kappa\gamma_t\beta + \kappa\beta_t\gamma), \\ I(z) &= \frac{1}{4}(u_x u_x - u_{xx}u - \kappa\beta_{xx}\beta + \kappa\beta_x\beta_x), \\ G(z) &= u\eta^2 - uv_x + \frac{1}{4}u_t\eta - \frac{1}{4}u\eta_t + \kappa\gamma\rho + \frac{\kappa}{4}\rho_t\beta - \frac{\kappa}{4}\beta_t\rho. \end{aligned}$$

As before, using the fact that w is constant in the variable x , η and v are periodic and integration by parts, the local conservation laws for 2HS give the following conserved quantities for travelling wave solutions

$$\mathcal{H}_1(u, u_x, \rho) = \frac{1}{2} \int (u_x^2 + \kappa\rho^2) dx, \quad \mathcal{H}_2(u, u_x, \rho) = \frac{1}{2} \int (\kappa u \rho^2 + uu_x^2) dx. \quad (28)$$

We can now derive, as it was done previously, the centered version of the explicit Euler box scheme (21) for (3)

$$\begin{aligned} -\delta_t u^{n,i} + (\delta_x^2)^\dagger \left(\frac{1}{2} \delta_x ((\delta_x u^{n,i})^2) - \delta_x^2 (u^{n,i} \delta_x u^{n,i}) + \frac{\kappa}{2} \delta_x ((\rho^{n,i})^2) \right) &= 0, \\ \delta_t \rho^{n,i} + \delta_x (u^{n,i} \rho^{n,i}) &= 0 \end{aligned} \quad (29)$$

with the pseudo-inverse operator $(\delta_x^2)^\dagger$.

Remark 2. Similarly as above, one can also find a multi-symplectic formulation of the generalised periodic two-component Hunter–Saxton system [26]

$$\begin{aligned} u_{txx} + 2\sigma u_x u_{xx} + \sigma u u_{xxx} - \kappa \rho \rho_x + A u_x &= 0, \\ \rho_t + (u\rho)_x &= 0. \end{aligned}$$

Here, in addition, $\sigma \in \mathbb{R}$ and $A \geq 0$. Indeed, setting $z = [u, \phi, w, v, \eta, \rho, \gamma, \beta]$, using the skew-symmetric matrices (27) and the gradient of the scalar function

$$S(z) = -wu - \sigma u \eta^2/2 + \eta v - \kappa u \rho^2/2 + \kappa \gamma \rho + A u^2/2$$

one obtains a multi-symplectic formulation (17) for the generalised Hunter–Saxton system (30). This would then permit to derive an Euler box scheme for these kind of equations too.

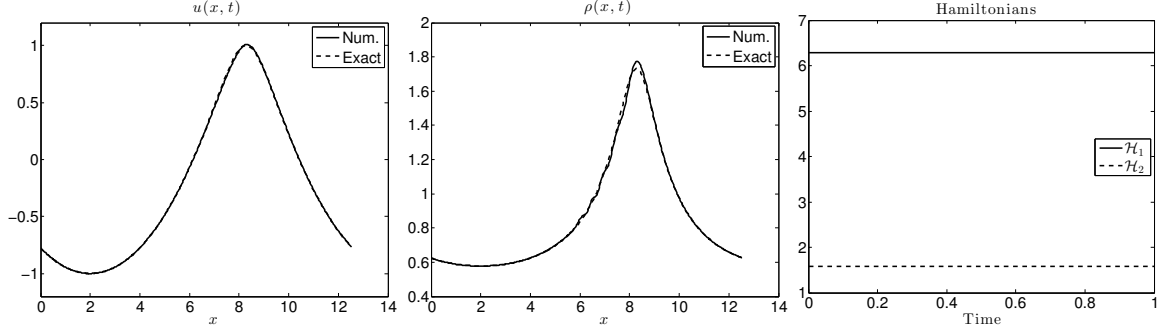


Figure 4: Exact and numerical profiles of $u(x, t)$ and $\rho(x, t)$ at time $T_{\text{end}} = 1$ and computed Hamiltonians (right plot) by the multi-symplectic scheme (29).

3.3.2 Multi-symplectic simulations of travelling waves for 2HS

We now test the multi-symplectic Euler box scheme (29) on travelling wave solutions of (3) with $\kappa = 1$. The smooth periodic travelling waves of speed c , $u(x, t) = \varphi(x - ct)$, are solutions to the differential equation (14), see also (15).

Figure 4 displays the exact and numerical profiles of $u(x, t)$ and $\rho(x, t)$ at time $T_{\text{end}} = 1$ and also the computed values of the Hamiltonians (28) using the multi-symplectic Euler box scheme (29) with (large) step sizes $\Delta t = 0.1$ and $\Delta x = L_{\text{per}}/512$. The parameters for this simulation on the periodic computational domain $[0, L_{\text{per}}]$ are as follows: $z = -1, Z = 1, b = 1, c = 2, a = \sqrt{3}$. We checked numerically that the solution of the differential equation (14) for $\varphi(0) = z$ and $\varphi'(0) = 0$ is periodic with period $L_{\text{per}} = 12.5663 \dots$. One can observe, in this figure, that the numerical profile of the solution agrees very well with the exact profile. Furthermore, excellent conservation properties of the numerical solution are monitored.

4 Hamiltonian-preserving discretisations of Hunter–Saxton-like equations

As seen in the previous section, the class of HS-like equations, considered here, possesses invariants. It is thus of interest to derive invariant-preserving numerical methods. Furthermore, further numerical methods are useful for comparison purposes as exact solutions are generally not available for such problems.

This section presents Hamiltonian-preserving schemes for the Hunter–Saxton equation, for the modified Hunter–Saxton equation as well as for the two-component Hunter–Saxton equation.

Though the derivation of the proposed Hamiltonian-preserving schemes basically follows the lines of [8], we need to pay attention on the treatment of boundary conditions.

4.1 Hamiltonian-preserving integrators for HS (half line case)

We first derive Hamiltonian-preserving numerical schemes for the HS equation based on the \mathcal{H}_1 -variational formulation, resp. \mathcal{H}_2 -variational formulation of the problem. In this subsection, we assume the boundary conditions given by $u(-L, t) = u_x(-L, t) = u_x(L, t) = u_{xx}(L, t) = 0$.

4.1.1 An integrator based on the \mathcal{H}_1 -variational formulation of HS

In this subsection, we propose a finite difference scheme which preserves the Hamiltonian of the following variational formulation of the HS equation [15]:

$$u_{xxt} = (u_{xx}\partial_x + \partial_x u_{xx})\partial_x^{-2} \frac{\delta \mathcal{H}_1}{\delta u}, \quad \mathcal{H}_1(u, u_x) = \frac{1}{2} \int_{-L}^L u_x^2 dx,$$

where $\partial_x^{-1}(\cdot) := \int_{-L}^x (\cdot) dx$. As in the usual interpretation, $(u_{xx}\partial_x + \partial_x u_{xx})$ operates on a function f such that $(u_{xx}\partial_x + \partial_x u_{xx})f = u_{xx}f_x + (u_{xx}f)_x$. Since $\delta\mathcal{H}_1/\delta u = -u_{xx}$, the above expression simplifies to

$$u_{xxt} = -(u_{xx}\partial_x + \partial_x u_{xx})u. \quad (31)$$

Under the boundary conditions $u(-L, t) = u_x(-L, t) = u_x(L, t) = u_{xx}(L, t) = 0$, we can prove the \mathcal{H}_1 -preservation based on (31). In fact, using the integration-by-parts formula and (31), we have

$$\begin{aligned} \frac{d}{dt}\mathcal{H}_1 &= \int_{-L}^L u_x u_{xt} dx = - \int_{-L}^L u u_{xxt} dx + [u u_{xt}]_{-L}^L \\ &= \int_{-L}^L u \cdot (u_{xx}\partial_x + \partial_x u_{xx})u dx = [u^2 u_{xx}]_{-L}^L = 0. \end{aligned}$$

Here, the boundary terms $[u u_{xt}]_{-L}^L$ and $[u^2 u_{xx}]_{-L}^L$ vanish due to the boundary conditions.

We now derive a finite difference scheme which preserves the Hamiltonian structure, namely, an \mathcal{H}_1 -preserving scheme. This derivation is based on the discrete variational derivative method [7, 8] (see also [4]). We focus on the spatial discretisation, since the idea of the temporal discretisation is essentially the same as the discrete gradient method (see, for example, [9, 29] and references therein).

Firstly, let us define a discrete version of \mathcal{H}_1 by

$$\mathcal{H}_{1,d}(\mathbf{u}) := \sum_{n=0}^N \frac{\Delta x}{2} \frac{(\delta_x^+ u^n)^2 + (\delta_x^- u^n)^2}{2}.$$

Based on the first Hamiltonian structure (31), we define the following semi-discrete scheme:

$$\tilde{\delta}_x^2 \dot{u}^n = -(\tilde{\delta}_x^2 u^n)(\delta_x u^n) - \delta_x(u^n(\tilde{\delta}_x^2 u^n)) \quad (32)$$

for $n = 1, \dots, N$, where the dot on u stands for the differentiation with respect to time. We consider the numerical boundary conditions $u^0 = 0$, $u^{-1} = u^1$, $u^{N+1} = u^{N-1}$, $u^{N+2} = 2u^N - u^{N-2}$, which correspond to $u(-L, t) = 0$, $u_x(-L, t) = 0$, $u_x(L, t) = 0$, $u_{xx}(L, t) = 0$, respectively. The semi-discrete scheme (32) inherits the \mathcal{H}_1 -preservation. To simplify the calculations of the proof, we extend the semi-discrete scheme to $n = 0$. Note that the scheme refers u^{-2} when $n = 0$, but we simply understand u^{-2} takes a value such that the scheme holds even for $n = 0$. The \mathcal{H}_1 -preservation is now checked as follows.

$$\begin{aligned} &\frac{d}{dt}\mathcal{H}_{1,d}(\mathbf{u}(t)) \\ &= \sum_{n=0}^N \frac{\Delta x}{2} ((\delta_x^+ u^n)(\delta_x^+ \dot{u}^n) + (\delta_x^- u^n)(\delta_x^- \dot{u}^n)) \\ &= - \sum_{n=0}^N \frac{\Delta x}{2} u^n (\tilde{\delta}_x^2 \dot{u}^n) \\ &\quad + \frac{1}{4} [(\delta_x^+ \dot{u}^n)u^{n+1} + (\delta_x^+ \dot{u}^{n-1})u^n + u^n(\delta_x^- \dot{u}^{n+1}) + u^{n-1}(\delta_x^- \dot{u}^n)]_0^N \\ &= \sum_{n=0}^N \frac{\Delta x}{2} \left(u^n (\tilde{\delta}_x^2 u^n)(\delta_x u^n) + u^n \delta_x(u^n(\tilde{\delta}_x^2 u^n)) \right) \\ &= \sum_{n=0}^N \frac{\Delta x}{2} \left(u^n (\tilde{\delta}_x^2 u^n)(\delta_x u^n) - (\delta_x u^n)u^n(\tilde{\delta}_x^2 u^n) \right) \\ &\quad + \frac{1}{4} \left[u^{n+1}(\tilde{\delta}_x^2 u^{n+1}) + u^{n-1}(\tilde{\delta}_x^2 u^{n-1}) + u^{n+1}(\tilde{\delta}_x^2 u^n) + u^{n-1}(\tilde{\delta}_x^2 u^n) \right]_0^N \\ &= 0. \end{aligned}$$

Next, we apply the discrete gradient method for the temporal discretisation in order to obtain the following fully-discrete scheme:

$$\tilde{\delta}_x^2 \delta_t^+ u^{n,i} = -(\tilde{\delta}_x^2 u^{n,i+1/2})(\delta_x u^{n,i+1/2}) - \delta_x(u^{n,i+1/2}(\tilde{\delta}_x^2 u^{n,i+1/2})) \quad (33)$$

for $n = 1, \dots, N$, still with the numerical boundary conditions $u^{0,i} = 0$, $u^{-1,i} = u^{1,i}$, $u^{N+1,i} = u^{N-1,i}$, $u^{N+2,i} = 2u^{N,i} - u^{N-2,i}$, which correspond to $u(-L, t) = 0$, $u_x(-L, t) = 0$, $u_x(L, t) = 0$, $u_{xx}(L, t) = 0$, respectively, for all i and $u^{n,i+1/2}$ denotes an abbreviation for $(u^{n,i} + u^{n,i+1})/2$. The scheme (33) has the following conservation property by construction

$$\mathcal{H}_{1,d}(\mathbf{u}^i) = \mathcal{H}_{1,d}(\mathbf{u}^0) \quad \text{for all } i.$$

Remark 3. The resulting scheme strongly depends on the definition of the discrete version of \mathcal{H}_1 . Other definitions such as, for example,

$$\mathcal{H}_{1,d}(\mathbf{u}) := \sum_{n=0}^N \frac{\Delta x}{2} (\delta_x u^n)^2$$

will lead to different \mathcal{H}_1 -preserving schemes.

4.1.2 An integrator based on the \mathcal{H}_2 -variational formulation of HS

In this subsection, we propose a finite difference scheme which preserves the second Hamiltonian structure of the HS equation [15]:

$$u_{xt} = \frac{\delta \mathcal{H}_2}{\delta u}, \quad \mathcal{H}_2(u, u_x) = \frac{1}{2} \int_{-L}^L u u_x^2 dx,$$

or equivalently

$$u_{xt} = -(uu_x)_x + \frac{u_x^2}{2}, \quad \text{or} \quad u_t = -uu_x + \partial_x^{-1} \left(\frac{u_x^2}{2} \right),$$

where $\partial_x^{-1}(\cdot) = \int_{-L}^x (\cdot) dx$. Note that \mathcal{H}_2 is not an invariant as already seen in Subsection 3.1. This can also be confirmed using the above variational structure. In fact, we have

$$\begin{aligned} \frac{d}{dt} \mathcal{H}_2 &= \int_{-L}^L \left(\frac{1}{2} u_x^2 u_t + uu_x u_{xt} \right) dx = \int_{-L}^L \left(\frac{1}{2} u_x^2 - (uu_x)_x \right) u_t dx + [uu_x u_t]_{-L}^L \\ &= \int_{-L}^L u_{tx} u_t dx = \left[\frac{1}{2} u_t^2 \right]_{-L}^L = \frac{1}{2} u_t^2 \Big|_{x=L}. \end{aligned} \quad (34)$$

Let us now define a discrete version of \mathcal{H}_2 by

$$\mathcal{H}_{2,d}(\mathbf{u}, \mathbf{v}) := \sum_{n=0}^N \frac{\Delta x}{2} u^n (v^n)^2,$$

where $v^n := \delta_x u^n$. Based on the second Hamiltonian structure, we define the following semi-discrete scheme:

$$\dot{u}^n = -u^n v^n + \delta_x^{-1} \left(\frac{1}{2} (v^n)^2 \right) \quad (35)$$

for $n = 1, \dots, N$. We assume the numerical boundary conditions $u^0 = 0$ and $v^N = 0$, which correspond to $u(-L, t) = 0$ and $u_x(L, t) = 0$, respectively, and $\delta_x^{-1} \left(\frac{1}{2} (v^n)^2 \right)$ is defined by

$$\delta_x^{-1} \left(\frac{1}{2} (v^n)^2 \right) = \begin{cases} 0, & \text{if } n = 0, 1, \\ 2\Delta x \sum_{k=1}^m \frac{v_{2k-1}^2}{2}, & \text{if } n = 2m \ (m \geq 1), \\ 2\Delta x \sum_{k=1}^m \frac{v_{2k}^2}{2}, & \text{if } n = 2m+1 \ (m \geq 1), \end{cases}$$

which is a proper discretisation of $\partial_x^{-1}(\frac{v^2}{2}) = \int_{-L}^x (\frac{v^2}{2}) dx$, and satisfies

$$\delta_x \delta_x^{-1} \left(\frac{1}{2} (v^n)^2 \right) = \frac{1}{2} (v^n)^2$$

for $n = 1, \dots, N$.

Next we define the additional condition $v^0 = 0$, which corresponds to $u_x(-L, t) = 0$, and calculate the temporal differentiation of $\mathcal{H}_{2,d}$. To simplify the calculation, we apply δ_x to (35) to obtain

$$\delta_x \dot{u}^n = -\delta_x (u^n v^n) + \frac{1}{2} (v^n)^2. \quad (36)$$

This relation holds for $n = 1, \dots, N-1$. We can now calculate $\frac{d}{dt} \mathcal{H}_{2,d}(\mathbf{u}(t), \mathbf{v}(t))$, which directly leads to the following result. In order to simplify the computations, we also assume $u^{-1} = u^1$, $v^{-1} = v^1$ and $u^{N+1} v^{N+1} = u^N v^N$ so that (36) also holds even for $n = 0, N$. Then the temporal differentiation of $\mathcal{H}_{2,d}$ is calculated as follows.

$$\begin{aligned} \frac{d}{dt} \mathcal{H}_{2,d}(\mathbf{u}(t), \mathbf{v}(t)) &= \sum_{n=0}^N {}''\Delta x \left(\frac{(v^n)^2}{2} \dot{u}^n + u^n v^n \dot{v}^n \right) \\ &= \sum_{n=0}^N {}''\Delta x \left(\frac{(v^n)^2}{2} - \delta_x (u^n v^n) \right) \dot{u}^n \\ &\quad + \left[\frac{1}{4} (u^n v^n (\dot{u}^{n+1} + \dot{u}^{n-1}) + (u^{n+1} v^{n+1} + u^{n-1} v^{n-1}) \dot{u}^n) \right]_0^N \\ &= \sum_{n=0}^N {}''\Delta x (\delta_x \dot{u}^n) \dot{u}^n + \frac{1}{4} (u^{N+1} v^{N+1} + u^{N-1} v^{N-1}) \dot{u}^N \\ &= \left[\frac{1}{4} (\dot{u}^{n+1} + \dot{u}^{n-1}) \dot{u}^n \right]_0^N + \frac{1}{4} (u^{N+1} v^{N+1} + u^{N-1} v^{N-1}) \dot{u}^N \\ &= \frac{1}{2} \dot{u}^{N-1} \dot{u}^N + \frac{1}{4} (u^{N+1} v^{N+1} + u^{N-1} v^{N-1}) \dot{u}^N \\ &= \frac{1}{2} \dot{u}^{N-1} \dot{u}^N + \frac{1}{2} u^{N-1} v^{N-1} \dot{u}^N. \end{aligned} \quad (37)$$

Note that the discrete property (37) is in good agreement with the continuous one (34), since v^{N-1} (in the last term) is close to 0 as will soon be observed in the numerical experiments. Applying the discrete gradient method for the temporal discretisation, we obtain the following fully-discrete scheme:

$$\delta_t^+ u^{n,i} = -u^{n,i+1/2} v^{n,i+1/2} + \delta_x^{-1} \left(\frac{(v^{n,i+1})^2 + (v^{n,i})^2}{4} \right) \quad (38)$$

for $n = 1, \dots, N$, still with the numerical boundary conditions $u^{0,i} = 0$ and $v^{N,i} = 0$, which correspond to $u(-L, t) = 0$ and $u_x(L, t) = 0$, respectively, for all i . By construction, the scheme (38) has the following property:

$$\begin{aligned} &\frac{1}{\Delta t} (\mathcal{H}_{2,d}(\mathbf{u}^{i+1}, \mathbf{v}^{i+1}) - \mathcal{H}_{2,d}(\mathbf{u}^i, \mathbf{v}^i)) \\ &= \frac{1}{2} (\delta_t^+ u^{N-1,i}) (\delta_t^+ u^{N,i}) + \frac{1}{2} u^{N-1,i+1/2} v^{N-1,i+1/2} (\delta_t^+ u^{N,i}). \end{aligned}$$

4.1.3 Numerical simulations: Hamiltonian-preserving schemes for HS on $[-L, L]$

We now test the above presented Hamiltonian-preserving integrators on the same problem as in Subsection 3.1.3.

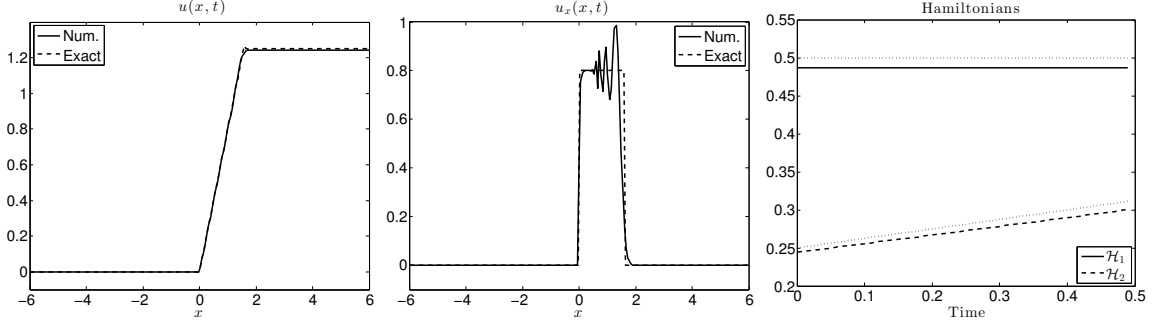


Figure 5: \mathcal{H}_1 -preserving scheme: exact and numerical profiles of $u(x, t)$ and $u_x(x, t)$ at time $T_{\text{end}} = 0.5$ and computed Hamiltonians ($\Delta x = 12/201$ and $\Delta t = 0.01$).

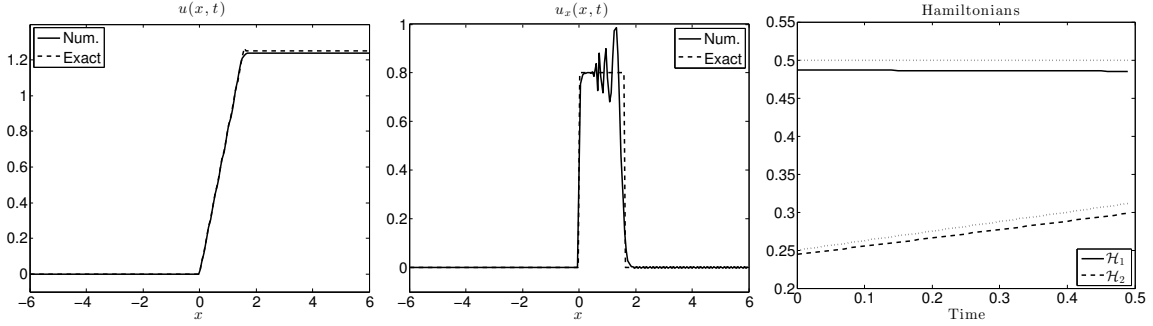


Figure 6: \mathcal{H}_2 -preserving scheme: exact and numerical profiles of $u(x, t)$ and $u_x(x, t)$ at time $T_{\text{end}} = 0.5$ and computed Hamiltonians ($\Delta x = 12/201$ and $\Delta t = 0.01$).

Figures 5 and 6 show the numerical results obtained by the \mathcal{H}_1 -preserving scheme (33) and \mathcal{H}_2 -preserving scheme (38) with step sizes $\Delta x = 12/201$ (i.e., $N = 201$) and $\Delta t = 0.01$. Despite the oscillations observed in $u_x(x, t)$, one can observe a correct numerical approximation of $u(x, t)$. Observe that the \mathcal{H}_1 -preserving scheme offers a good behaviour in terms of the \mathcal{H}_2 -quantity. For the \mathcal{H}_2 -preserving scheme, though the slope of \mathcal{H}_2 does not perfectly coincide with that of the exact solution, the numerical result almost coincides with the one from the \mathcal{H}_1 -preserving scheme. Furthermore, we also observe a good behaviour in terms of the \mathcal{H}_1 -quantity. From a viewpoint of qualitative behaviour, no notable differences of the numerical solutions are observed for these two different energy-preserving schemes.

4.2 Hamiltonian-preserving integrator for the modified Hunter–Saxton equation (periodic case)

In this subsection, we propose an \mathcal{H}_1 -preserving finite difference scheme for the mHS equation. In order to derive the scheme, we use the first Hamiltonian structure [20]:

$$u_{xxt} = ((u_{xx} - \omega)\partial_x + \partial_x(u_{xx} - \omega))\partial_x^{-2} \frac{\delta \mathcal{H}_1}{\delta u}, \quad \mathcal{H}_1(u, u_x) = \frac{1}{2} \int_0^L u_x^2 dx,$$

which can be expressed more explicitly as

$$u_{xxt} = -(u_{xx}\partial_x + \partial_x u_{xx})u + 2\omega u_x.$$

In fact, one can prove the \mathcal{H}_1 -preservation (25) as follows:

$$\frac{d}{dt} \mathcal{H}_1 = \int_0^L u_x u_{xt} dx = - \int_0^L u u_{xxt} dx = \int_0^L u \cdot (u_{xx}\partial_x + \partial_x u_{xx})u dx - 2\omega \int_0^L u u_x dx = 0. \quad (39)$$

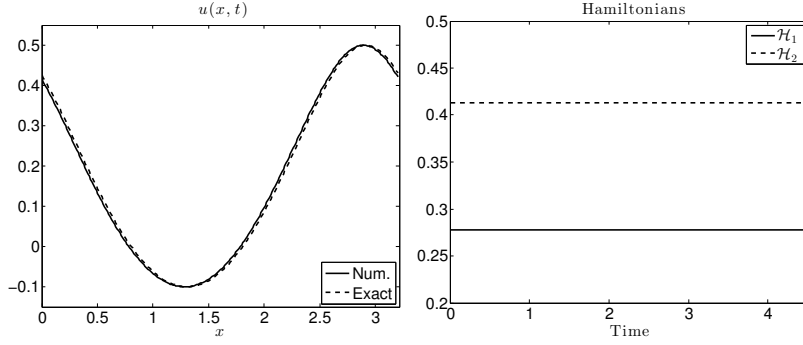


Figure 7: Exact and numerical profiles of $u(x, t)$ at time $T_{\text{end}} = 3.5$ and computed Hamiltonians by the \mathcal{H}_1 -preserving scheme (41).

Here, no boundary terms appear due to the periodicity of solutions. The last equality follows from the skew-symmetry of $(u_{xx}\partial_x + \partial_x u_{xx})$ and ∂_x .

Based on this formulation, one derives the following \mathcal{H}_1 -preserving scheme

$$\tilde{\delta}_x^2 \delta_t^+ u^{n,i} + (\tilde{\delta}_x^2 u^{n,i+1/2})(\delta_x u^{n,i+1/2}) + \delta_x(u^{n,i+1/2}(\tilde{\delta}_x^2 u^{n,i+1/2})) - 2\omega \delta_x u^{n,i+1/2} = 0, \quad (40)$$

for which the solutions satisfy

$$\mathcal{H}_{1,d}(\mathbf{u}^i) = \mathcal{H}_{1,d}(\mathbf{u}^0) \quad \text{for all } i,$$

where

$$\mathcal{H}_{1,d}(\mathbf{u}^i) := \sum_{n=0}^{N-1} \Delta x \frac{(\delta_x^+ u^n)^2}{2}.$$

This conservation property is checked by calculations corresponding to (39) in the discrete setting.

As this was done for the multi-symplectic integrator, in order to select the right travelling waves, we have to consider the pseudo-inverse operator of $\tilde{\delta}_x^2$, denoted by $(\tilde{\delta}_x^2)^\dagger$. We then obtain the following scheme

$$\delta_t^+ u^{n,i} + (\tilde{\delta}_x^2)^\dagger \left((\tilde{\delta}_x^2 u^{n,i+1/2})(\delta_x u^{n,i+1/2}) + \delta_x(u^{n,i+1/2}(\tilde{\delta}_x^2 u^{n,i+1/2})) - 2\omega \delta_x u^{n,i+1/2} \right) = 0. \quad (41)$$

The above conservation property remains since the solution of (41) always satisfies (40).

We now apply the scheme (41) to the same problem as in Subsection 3.2.2. Figure 7 displays the exact and numerical profiles of $u(x, t)$ at time $T_{\text{end}} = 3.5$ and also the computed values of the Hamiltonians

$$\mathcal{H}_1(u, u_x) = \frac{1}{2} \int_0^L u_x^2 dx, \quad \mathcal{H}_2(u, u_x) = \frac{1}{2} \int_0^L (uu_x^2 + 2\omega u^2) dx$$

with (relative large) step sizes $\Delta t = 0.02$ and $\Delta x = L_{\text{per}}/256$.

One may notice that the numerical solutions given by this integrator agree very well with the exact ones as well as with the ones given by the multi-symplectic schemes from the previous section. In addition to the exact preservation of \mathcal{H}_1 , the quantity \mathcal{H}_2 is also preserved with a very good accuracy.

4.3 Hamiltonian-preserving integrator for the two component Hunter–Saxton equation (periodic case)

In this subsection, we propose an \mathcal{H}_1 -preserving finite difference scheme for the 2HS equation. In order to derive the scheme, we use the first Hamiltonian structure [21]:

$$\begin{bmatrix} u_{xxt} \\ \rho_t \end{bmatrix} = \begin{bmatrix} (u_{xx}\partial_x + \partial_x u_{xx}) & \rho \partial_x \\ \partial_x \rho & 0 \end{bmatrix} \begin{bmatrix} \partial_x^{-2} \delta \mathcal{H}_1 / \delta u \\ \delta \mathcal{H}_1 / \delta \rho \end{bmatrix}, \quad \mathcal{H}_1(u, u_x, \rho) = \frac{1}{2} \int (u_x^2 + \kappa \rho^2) dx,$$

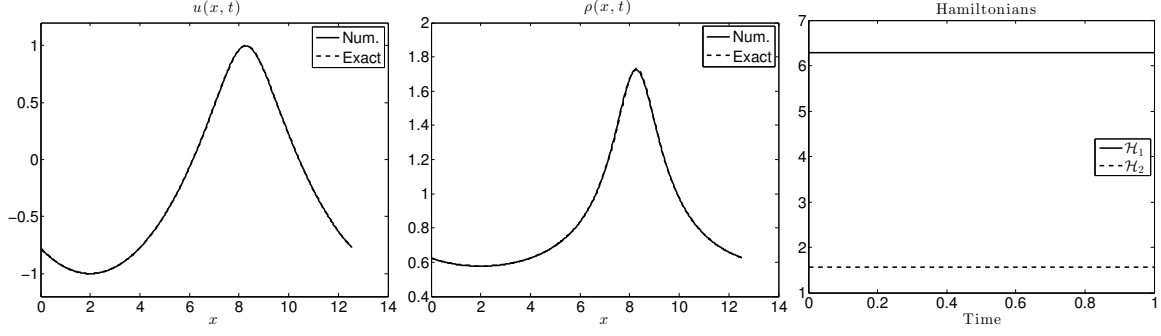


Figure 8: Exact and numerical profiles of $u(x, t)$ and $\rho(x, t)$ at time $T_{\text{end}} = 1$ and computed Hamiltonians (right plot) by the \mathcal{H}_1 -preserving scheme (42).

which can be expressed more explicitly as

$$\begin{aligned} u_{xxt} &= -(u_{xx}\partial_x + \partial_x u_{xx})u + \kappa\rho\rho_x, \\ \rho_t &= -(u\rho)_x. \end{aligned}$$

In fact, one can prove the \mathcal{H}_1 -preservation (28) as follows:

$$\begin{aligned} \frac{d}{dt}\mathcal{H}_1 &= \int_0^L u_x u_{xt} dx + \kappa \int_0^L \rho \rho_t dx = - \int_0^L uu_{xxt} dx - \kappa \int_0^L \rho(u\rho)_x dx \\ &= \int_0^L u \cdot (u_{xx}\partial_x + \partial_x u_{xx})u dx - \kappa \int_0^L u\rho\rho_x dx - \kappa \int_0^L \rho(u\rho)_x dx \\ &= -\kappa \int_0^L u\rho\rho_x dx + \kappa \int_0^L \rho_x u\rho dx = 0. \end{aligned}$$

As it was done previously, one derives the following \mathcal{H}_1 -preserving scheme

$$\begin{aligned} \delta_t^+ u^{n,i} + (\tilde{\delta}_x^2)^\dagger \left((\tilde{\delta}_x^2 u^{n,i+1/2})(\delta_x u^{n,i+1/2}) + \delta_x(u^{n,i+1/2}(\tilde{\delta}_x^2 u^{n,i+1/2})) - \kappa \rho^{n,i+1/2} \delta_x \rho^{n,i+1/2} \right) &= 0, \\ \delta_t^+ \rho^{n,i} + \delta_x(u^{n,i+1/2} \rho^{n,i+1/2}) &= 0, \end{aligned}$$

with the pseudo-inverse operator $(\tilde{\delta}_x^2)^\dagger$. This scheme possesses a conservation property

$$\mathcal{H}_{1,d}(\mathbf{u}^i) = \mathcal{H}_{1,d}(\mathbf{u}^0) \quad \text{for all } i,$$

where

$$\mathcal{H}_{1,d}(\mathbf{u}^i) := \sum_{n=0}^{N-1} \Delta x \frac{(\delta_x^+ u^n)^2 + \kappa(\rho^n)^2}{2}.$$

We now test the \mathcal{H}_1 -preserving scheme (42) on the travelling wave solutions of (3) with $\kappa = 1$ used in Subsection 3.3.2.

Figure 8 displays the exact and numerical profiles of $u(x, t)$ and $\rho(x, t)$ at time $T_{\text{end}} = 1$ and also the computed values of the Hamiltonians (28) using the \mathcal{H}_1 -preserving scheme (42) with (large) step sizes $\Delta t = 0.1$ and $\Delta x = L_{\text{per}}/512$.

The numerical profiles of the solutions agree very well with the exact profiles as well as with the profiles of the multi-symplectic scheme displayed in Figure 4. Further, excellent conservation properties of the numerical solutions are observed.

5 Concluding remarks

This study contributes to enhance multi-symplectic discretisations for partial differential equations arising from important applications in physics. Indeed, we have presented the first multi-symplectic formulations of the Hunter–Saxton equation, of the modified Hunter–Saxton equation and of the two component Hunter–Saxton system. Furthermore, using these results, we have derived novel explicit multi-symplectic integrators for these nonlinear partial differential equations.

As exact solutions to these PDEs are rarely known, further numerical methods are useful for comparison. We therefore investigate Hamiltonian-preserving numerical discretisations for these problems. No significant differences between this type of geometric numerical schemes and the multi-symplectic integrators are observed except, perhaps, the fact that the last one seem a little bit better from a practical viewpoint, being explicit and a little easier to implement.

A major difficulty in the derivation of numerical schemes for the HS-like equations is a proper treatment of the boundary conditions. We clarify this issue for the exact as well as for the numerical solutions to these PDEs. Therefore, all the numerical methods proposed in this publication enjoy a correct treatment of the boundary conditions for the HS-like equations.

So far, numerical tests have been conducted only with the Euler box scheme. Besides numerical experiments have been conducted for particular travelling wave solutions of the modified Hunter–Saxton equation or the two component Hunter–Saxton system. It thus remains to try out and analyse more elaborate structure-preserving numerical methods and other type of solutions to these problems.

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